

Quiz 3 - Variant 1 - Solutions
CRN 10895

Question 1 (1.5 points). Find the measures of the angles of the triangle whose vertices are

$$A = (-1, 0), \quad B = (2, 1), \quad C = (1, -2).$$

Solution We use the dot-product formula

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

At vertex A the side vectors are

$$\overrightarrow{AB} = (3, 1), \quad \overrightarrow{AC} = (2, -2).$$

Hence

$$\cos A = \frac{(3)(2) + (1)(-2)}{\sqrt{10}\sqrt{8}} = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}.$$

At vertex B , use $\overrightarrow{BA} = (-3, -1)$ and $\overrightarrow{BC} = (-1, -3)$. Then

$$\cos B = \frac{(-3)(-1) + (-1)(-3)}{\sqrt{10}\sqrt{10}} = \frac{6}{10} = \frac{3}{5}.$$

At vertex C , use $\overrightarrow{CA} = (-2, 2)$ and $\overrightarrow{CB} = (1, 3)$, so

$$\cos C = \frac{(-2)(1) + (2)(3)}{\sqrt{8}\sqrt{10}} = \frac{1}{\sqrt{5}}.$$

Therefore the three angles (in degrees) are

$$A = C = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435^\circ, \quad B = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130^\circ.$$

Their sum is 180° , as required.

Question 2 (1.5 points). Write inequalities describing (a) the interior and (b) the exterior of the sphere of radius 1 centered at $(1, 1, 1)$.

Solution The squared distance from (x, y, z) to the center $(1, 1, 1)$ is

$$d^2 = (x - 1)^2 + (y - 1)^2 + (z - 1)^2.$$

The interior consists of points whose distance from the center is strictly less than 1. Hence

$$(a) \quad (x - 1)^2 + (y - 1)^2 + (z - 1)^2 < 1.$$

The exterior consists of points whose distance is strictly greater than 1. Thus

$$(b) \quad (x - 1)^2 + (y - 1)^2 + (z - 1)^2 > 1.$$

Equality describes the sphere itself and is excluded from both the interior and the exterior.

Quiz 3 - Variant 2 - Solutions
CRN 10895

Question 1 (1.5 points). Find the measures of the angles between the diagonals of the rectangle whose vertices are

$$A = (1, 0), \quad B = (0, 3), \quad C = (3, 4), \quad D = (4, 1).$$

Solution The diagonals are AC and BD . Their direction vectors are

$$\overrightarrow{AC} = C - A = (2, 4), \quad \overrightarrow{BD} = D - B = (4, -2).$$

Their dot product equals

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = (2)(4) + (4)(-2) = 8 - 8 = 0.$$

Since both vectors are nonzero, the dot-product formula gives

$$\cos \theta = \frac{0}{\sqrt{20}\sqrt{20}} = 0.$$

Thus the two diagonals are perpendicular. The angles formed by their intersection are

$\theta = 90^\circ \quad (\text{all four angles}).$

Question 2 (1.5 points). Write inequalities describing the closed region bounded by the spheres of radii 1 and 2 centered at the origin.

Solution For a point (x, y, z) , the distance to the origin is

$$r = \sqrt{x^2 + y^2 + z^2}.$$

The required region lies outside the sphere of radius 1 and inside the sphere of radius 2. Because the region is closed, points on both spheres are included. Therefore,

$$1 \leq r \leq 2.$$

Since $r \geq 0$, squaring yields the equivalent description

$1 \leq x^2 + y^2 + z^2 \leq 4.$

Quiz 3 - Variant 1 - Solutions
CRN 10896

Question 1 (1.5 points). Find the volume of the parallelepiped determined by

$$\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{w} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Solution The volume of a parallelepiped determined by three vectors is the absolute value of their scalar triple product:

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

First compute the cross product:

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} \\ &= (1 \cdot (-1) - (-2) \cdot 2)\mathbf{i} - (2 \cdot (-1) - (-2) \cdot (-1))\mathbf{j} + (2 \cdot 2 - 1 \cdot (-1))\mathbf{k} \\ &= 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}. \end{aligned}$$

Taking the dot product with $\mathbf{u} = (1, -1, 1)$,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1(3) + (-1)(4) + 1(5) = 4.$$

Therefore the volume is

$V = |4| = 4.$

Question 2 (1.5 points). Find the center and radius of the sphere

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9.$$

Solution Dividing by 2 and grouping terms gives

$$x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2}.$$

Completing the square in each variable,

$$\begin{aligned} (x + \frac{1}{4})^2 + (y + \frac{1}{4})^2 + (z + \frac{1}{4})^2 &= \frac{9}{2} + 3\left(\frac{1}{16}\right) \\ &= \frac{75}{16}. \end{aligned}$$

Comparing with $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$, we obtain

Center $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right), \quad r = \frac{5\sqrt{3}}{4}.$

Quiz 3 - Variant 2 - Solutions
CRN 10896

Question 1 (1.5 points). Find the volume of the parallelepiped determined by

$$\mathbf{u} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{v} = -\mathbf{i} - \mathbf{k}, \quad \mathbf{w} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

Solution Using the scalar triple product,

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

We first calculate

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} \\ &= (0 \cdot (-2) - (-1) \cdot 4)\mathbf{i} - ((-1)(-2) - (-1)(2))\mathbf{j} + ((-1)(4) - 0(2))\mathbf{k} \\ &= 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1(4) + 1(-4) + (-2)(-4) = 8.$$

Taking the absolute value gives

$$\boxed{V = 8.}$$

Question 2 (1.5 points). Find the center and radius of the sphere

$$3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9.$$

Solution Dividing by 3 gives

$$x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3.$$

Completing the square in y and z ,

$$\begin{aligned} x^2 + (y + \frac{1}{3})^2 + (z - \frac{1}{3})^2 &= 3 + \frac{1}{9} + \frac{1}{9} \\ &= \frac{29}{9}. \end{aligned}$$

Hence

$$\boxed{\text{Center } \left(0, -\frac{1}{3}, \frac{1}{3}\right), \quad r = \frac{\sqrt{29}}{3}.$$

Quiz 3 - Variant 1 - Solutions
CRN 10897

Question 1 (1.5 points). Find the area of triangle PQR and a unit vector perpendicular to plane PQR , where

$$P = (1, -1, 2), \quad Q = (2, 0, -1), \quad R = (0, 2, 1).$$

Solution The two side vectors starting at P are

$$\overrightarrow{PQ} = Q - P = (1, 1, -3), \quad \overrightarrow{PR} = R - P = (-1, 3, -1).$$

Their cross product is

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} \\ &= (8, 4, 4). \end{aligned}$$

The area is half the magnitude of the cross product:

$$A = \frac{1}{2} \sqrt{8^2 + 4^2 + 4^2} = \frac{1}{2} \sqrt{96} = 2\sqrt{6}.$$

A perpendicular unit vector is the normalized cross product:

$$\frac{(8, 4, 4)}{\sqrt{96}} = \frac{(2, 1, 1)}{\sqrt{6}}.$$

Therefore

$$A = 2\sqrt{6}, \quad \mathbf{n} = \pm \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Either sign is acceptable because both vectors are perpendicular to plane PQR .

Question 2 (1.5 points). Find the vector projection $\text{proj}_{\mathbf{v}} \mathbf{u}$, where

$$\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}, \quad \mathbf{u} = 3\mathbf{j} + 4\mathbf{k}.$$

Solution The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

Here $\mathbf{u} = (0, 3, 4)$ and $\mathbf{v} = (10, 11, -2)$. We calculate

$$\mathbf{u} \cdot \mathbf{v} = 0(10) + 3(11) + 4(-2) = 25,$$

$$\|\mathbf{v}\|^2 = 10^2 + 11^2 + (-2)^2 = 225.$$

Consequently,

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{25}{225} (10, 11, -2) = \frac{10}{9} \mathbf{i} + \frac{11}{9} \mathbf{j} - \frac{2}{9} \mathbf{k}.$$

Quiz 3 - Variant 2 - Solutions
CRN 10897

Question 1 (1.5 points). Find the area of triangle PQR and a unit vector perpendicular to plane PQR , where

$$P = (1, 1, 1), \quad Q = (2, 1, 3), \quad R = (3, -1, 1).$$

Solution The vectors along two sides starting from P are

$$\overrightarrow{PQ} = (1, 0, 2), \quad \overrightarrow{PR} = (2, -2, 0).$$

Their cross product is

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} \\ &= (4, 4, -2). \end{aligned}$$

Its magnitude equals

$$\sqrt{4^2 + 4^2 + (-2)^2} = \sqrt{36} = 6.$$

The area of the triangle is half this magnitude, so $A = 3$. Normalizing the cross product gives

$$\mathbf{n} = \pm \frac{(4, 4, -2)}{6} = \pm \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right).$$

Therefore,

$$A = 3, \quad \mathbf{n} = \pm \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}).$$

Question 2 (1.5 points). Find the vector projection $\text{proj}_{\mathbf{v}} \mathbf{u}$, where

$$\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}, \quad \mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Solution Recall that

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

For $\mathbf{u} = (2, 2, 1)$ and $\mathbf{v} = (2, 10, -11)$,

$$\mathbf{u} \cdot \mathbf{v} = 2(2) + 2(10) + 1(-11) = 13,$$

$$\mathbf{v} \cdot \mathbf{v} = 2^2 + 10^2 + (-11)^2 = 225.$$

Thus

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{13}{225}(2, 10, -11) = \frac{26}{225}\mathbf{i} + \frac{26}{45}\mathbf{j} - \frac{143}{225}\mathbf{k}.$$