

Calculus 2 - Quiz 2

Exercise 1

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Step 1: 0.5pts

Continuous: $f(x)$ is continuous on $[2, +\infty)$ because there are no divisions by zero or negative ~~base~~ logarithms for $x \geq 2$.

Positive: For all $x \geq 2$, $x > 0$ and $\ln x > \ln 2$, so $f(x) > 0$

Decreasing: As x increases, both x and $(\ln x)^2$ increase, which means the denominator increases and overall fraction $f(x)$ decreases

Step 2: 0.5pts

Evaluate corresponding improper integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx = \left[-\frac{1}{\ln x} \right]_2^t = -\frac{1}{\ln t} - \left(-\frac{1}{\ln 2} \right) =$$

$$\int \frac{1}{x(\ln x)^2} dx = \int \frac{1}{u^2} du = -\frac{1}{u} = -\frac{1}{\ln x} \quad \begin{matrix} u = \ln x \\ du = \frac{1}{x} dx \end{matrix}$$

$$\lim_{t \rightarrow \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln t} \right) = \frac{1}{\ln 2}$$

Conclusion 0.5pts

Since improper integral converges to finite value $\left(\frac{1}{\ln 2} \right)$ the series converges by the integral test.

Exercise 2

Step 1: Choose ~~comp~~ comparison series 0.5pts

$$a_n = \frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}} \approx \frac{\sqrt{n}}{\sqrt{n^2}} = \frac{\sqrt{n}}{n} = \frac{1}{n^{\frac{1}{2}}}$$

divergent p-series

$$p = \frac{1}{2} < 1$$

b_n diverges

Step 2: Compute limit.

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}(\sqrt{n} + 1)}{\sqrt{n^2 + 3}} = 1$$

$$\lim_{n \rightarrow \infty} \frac{n + \sqrt{n}}{\sqrt{n^2 + 3}} = \lim_{n \rightarrow \infty} \frac{n(1 + \frac{1}{\sqrt{n}})}{n\sqrt{1 + \frac{3}{n^2}}} = 1$$

Conclusion (0.5pt)

Since L is finite, positive both series share same behavior by limit comparison

Test. Because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series diverges

Calculus 2 - Quiz 2

Exercise 1

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

steps: check conditions of integral test (0.5 pt)

let $f(x) = \frac{1}{x^2+4}$ for $x \geq 1$

Positive: $f(x) > 0$ for all $x \geq 1$

numerator and denominator are both strictly positive

Continuous: $f(x)$ is continuous on $[1, \infty)$ since $x^2+4 > 0$ never equals to 0

Decreasing: Using the quotient rule

$$f'(x) = \frac{4-x^2}{(x^2+4)^2}$$

which is negative for $x > 2 \Rightarrow$ function is decreasing.

0.5 pt Step 2: set up and evaluate the improper integral

$$\int_1^{\infty} \frac{1}{x^2+4} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+4} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{x}{u} \frac{du}{2x} =$$

$$u = x^2+4$$

$$du = 2x dx$$

$$\lim_{t \rightarrow \infty} \int_1^t \frac{1}{2u} du =$$

$$\frac{1}{2} \lim_{t \rightarrow \infty} \int_1^t \frac{1}{u} du =$$

$$\frac{1}{2} \lim_{t \rightarrow \infty} \ln|t|$$

as $t \rightarrow \infty$, which gives ∞

$$\int \frac{1}{u} du = \ln|u| + C$$

$$\ln|u| \Big|_1^t =$$

$$\ln|t| - 0$$

0.5 pt Step 3:

since improper integral diverges to infinity, series diverges by integral test.

Question 2

$$\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$$

Step 1: Choose the Comparison series (0.5pt)

$$b_n = \frac{1}{n^{\frac{3}{2}}} \text{ (a convergent } p\text{-series where } p = \frac{3}{2} > 1)$$

Step 2: Apply the limit comparison test (0.5pt)

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\frac{n(1+\frac{4}{n})}{n^4(1+\frac{4}{n^4})}}}{\frac{1}{n^{\frac{3}{2}}}} = \frac{\lim_{n \rightarrow \infty} \sqrt{\frac{1}{n^3}}}{\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{3}{2}}}} = \frac{\frac{1}{n^{\frac{3}{2}}}}{\frac{1}{n^{\frac{3}{2}}}} = 1 \text{ Converges}$$

Step 3: Since limit is 1 (a finite, positive number, $0 < 1 < \infty$), both series behave same way.

Because $\sum \frac{1}{n^{\frac{3}{2}}}$ converges, the given series converges