

Quiz 2 - Variant 1 - Solutions
CRN 10895

Question 1 (1.5 points). Determine whether the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}.$$

Solution

Taking absolute values, we obtain

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n+1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n^2} \right).$$

Since $\frac{1}{n} + \frac{1}{n^2} \geq \frac{1}{n}$ for every $n \geq 1$, and the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, the Comparison Test shows that the series of absolute values diverges. Hence the given series is *not absolutely convergent*.

To apply the Alternating Series Test, let

$$b_n = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}.$$

We have $b_n > 0$ for every $n \geq 1$. Both $1/n$ and $1/n^2$ decrease as n increases, so $b_{n+1} < b_n$. $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) = 0$. Therefore the Alternating Series Test proves that the original series converges. Because it does not converge absolutely, its convergence is conditional.

Therefore, The series converges conditionally.

Question 2 (1.5 points). Find the Maclaurin series for the following function.

$$f(x) = \frac{2+x}{1-x}.$$

Solution

The Maclaurin series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Rewrite $f(x) = 3(1-x)^{-1} - 1$. Thus $f(0) = 2$. For every integer $n \geq 1$, repeated differentiation gives

$$f^{(n)}(x) = 3n!(1-x)^{-n-1}, \quad f^{(n)}(0) = 3n!.$$

Using the formula we obtain

$$2 + 3x + 3x^2 + 3x^3 + \cdots = 2 + 3 \sum_{n=1}^{\infty} x^n.$$

Quiz 2 - Variant 2 - Solutions
CRN 10895

Question 1 (1.5 points). Determine whether the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+3}.$$

Solution

Taking absolute values, we have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n+3} \right| = \sum_{n=1}^{\infty} \frac{1}{n+3} = \sum_{k=4}^{\infty} \frac{1}{k}.$$

This is the harmonic series with its first three terms removed. Omitting finitely many terms does not change convergence, so the absolute-value series diverges.

To apply the Alternating Series Test, put $b_n = 1/(n+3)$. We have $b_n > 0$ for every $n \geq 1$. $b_{n+1} = \frac{1}{n+4} < \frac{1}{n+3} = b_n$. $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n+3} = 0$. Thus the alternating series converges. Since it fails the absolute-convergence test, it converges conditionally.

Therefore, The series converges conditionally.

Question 2 (1.5 points). Find the Maclaurin series for the following function.

$$f(x) = \frac{x^2}{x+1}.$$

Solution

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Recall the geometric series:

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \quad |r| < 1.$$

Taking $r = -x$, we obtain

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1.$$

Multiplying both sides by x^2 , we obtain the Maclaurin series of $f(x) = \frac{x^2}{1+x}$:

$$x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2} = x^2 - x^3 + x^4 - x^5 + \cdots.$$

Quiz 2 - Variant 1 - Solutions
CRN 10896

Question 1 (1.5 points). Find the radius and interval of convergence of the following power series, identifying where convergence is absolute or conditional.

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}.$$

Solution

By the Ratio Test, With $a_n = (3x-2)^n/n$ and $x \neq 2/3$,

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \right| \\ &= |3x-2| \frac{n}{n+1} \rightarrow |3x-2|. \end{aligned}$$

At $x = 2/3$, every term is zero and convergence is immediate. The Ratio Test guarantees absolute convergence if $|3x-2| < 1$. Solve:

$$-1 < 3x-2 < 1 \iff 1 < 3x < 3 \iff \frac{1}{3} < x < 1.$$

Writing $3x-2 = 3(x-2/3)$ shows the center is $c = 2/3$ and the radius is $R = 1/3$.

At $x = 1/3$, $3x-2 = -1$, so the series becomes $\sum_{n=1}^{\infty} (-1)^n/n$. *First*, its absolute-value series $\sum_{n=1}^{\infty} 1/n$ diverges (harmonic series). *Next*, for $b_n = 1/n$, we have $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. By the Alternating Series Test, the original series converges. Thus convergence at this endpoint is conditional.

At $x = 1$, $3x-2 = 1$, so the series is the divergent harmonic series $\sum 1/n$.

Therefore, the interval of convergence is $[1/3, 1)$; absolute convergence on $(1/3, 1)$ and conditional convergence at $x = 1/3$.

Question 2 (1.5 points). Find the Taylor series generated by the following function at the indicated center.

$$f(x) = \frac{1}{x^2}, \quad a = 1.$$

Solution

The Taylor series at $a = 1$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n.$$

For $f(x) = x^{-2}$, the first few derivatives are

$$f'(x) = -2x^{-3}, \quad f''(x) = 6x^{-4}, \quad f^{(3)}(x) = -24x^{-5}.$$

The general formula is

$$f^{(n)}(x) = (-1)^n (n+1)! x^{-n-2}.$$

Consequently, $f^{(n)}(1) = (-1)^n (n+1)!$.

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} (x-1)^n &= \sum_{n=0}^{\infty} (-1)^n (n+1) (x-1)^n \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots \end{aligned}$$

Quiz 2 - Variant 2 - Solutions
CRN 10896

Question 1 (1.5 points). Find the radius and interval of convergence of the following power series, identifying where convergence is absolute or conditional.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}.$$

Solution

By the Ratio Test, For $a_n = (-1)^n (x+2)^n / n$ (away from $x = -2$),

$$\left| \frac{a_{n+1}}{a_n} \right| = |x+2| \frac{n}{n+1} \rightarrow |x+2|.$$

At $x = -2$, all terms equal zero. Absolute convergence occurs when

$$|x+2| < 1 \iff -1 < x+2 < 1 \iff -3 < x < -1.$$

The center is $c = -2$ and the radius is $\boxed{R = 1}$.

At $x = -3$, Here $x+2 = -1$, and

$$\frac{(-1)^n (-1)^n}{n} = \frac{1}{n}.$$

The resulting harmonic series diverges.

At $x = -1$, Here $x+2 = 1$, so the series becomes $\sum (-1)^n / n$. It converges by the Alternating Series Test, while its absolute-value series $\sum 1/n$ diverges. Thus convergence is conditional.

Therefore, the interval of convergence is $\boxed{(-3, -1]}$; absolute convergence on $(-3, -1)$, conditional convergence at $x = -1$.

Question 2 (1.5 points). Find the Taylor series generated by the following function at the indicated center.

$$f(x) = \frac{1}{(1-x)^3}, \quad a = 0.$$

Solution

The Maclaurin series formula is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

For $f(x) = (1-x)^{-3}$, repeated differentiation gives

$$\begin{aligned} f'(x) &= 3(1-x)^{-4}, \\ f''(x) &= 3 \cdot 4(1-x)^{-5}, \\ f^{(3)}(x) &= 3 \cdot 4 \cdot 5(1-x)^{-6}. \end{aligned}$$

Thus, for all $n \geq 0$,

$$f^{(n)}(x) = \frac{(n+2)!}{2!} (1-x)^{-n-3}, \quad f^{(n)}(0) = \frac{(n+2)!}{2}.$$

Hence, the coefficients are

$$\frac{f^{(n)}(0)}{n!} = \frac{(n+2)!}{2n!} = \frac{(n+1)(n+2)}{2}.$$

Therefore

$$\boxed{\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n = 1 + 3x + 6x^2 + 10x^3 + \cdots}$$

Quiz 2 - Variant 1 - Solutions
CRN 10897

Question 1. Determine whether the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} (-1)^n n^2 \left(\frac{1}{3}\right)^n.$$

Solution

Since $|(-1)^n| = 1$,

$$\sum_{n=1}^{\infty} \left| (-1)^n n^2 \left(\frac{1}{3}\right)^n \right| = \sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

Apply the Ratio Test to $b_n = n^2/3^n > 0$:

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{(n+1)^2}{3^{n+1}} \frac{3^n}{n^2} = \frac{1}{3} \left(\frac{n+1}{n} \right)^2 \\ &= \frac{1}{3} \left(1 + \frac{1}{n} \right)^2 \rightarrow \frac{1}{3} < 1. \end{aligned}$$

By the Ratio Test, the series of absolute values converges.

Therefore, The original series converges absolutely.

Question 2. Find the first three nonzero terms of the Maclaurin series for the following function and the values of x for which the series converges absolutely.

$$f(x) = \cos x - \frac{3}{1-x}.$$

Solution

Using the Maclaurin formula,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The required derivatives are

$$\begin{aligned} f(x) &= \cos x - 3(1-x)^{-1}, \\ f'(x) &= -\sin x - 3(1-x)^{-2}, \\ f''(x) &= -\cos x - 6(1-x)^{-3}. \end{aligned}$$

Evaluating at $x = 0$,

$$f(0) = 1 - 3 = -2, \quad f'(0) = -3, \quad f''(0) = -1 - 6 = -7.$$

Series.

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots \\ &= \boxed{-2 - 3x - \frac{7}{2}x^2 + \cdots}. \end{aligned}$$

These are the first three nonzero terms.

The Maclaurin series for $\cos x$ converges absolutely for every real x . The Maclaurin series for $\frac{1}{1-x}$, which is geometric series, converges absolutely on $(-1, 1)$.

Therefore, the series converges absolutely exactly for $-1 < x < 1$.

Quiz 2 - Variant 2 - Solutions
CRN 10897

Question 1. Determine whether the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-20)^n}{n!}.$$

Solution

Taking absolute values, we have

$$\sum_{n=1}^{\infty} \left| \frac{(-20)^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{20^n}{n!}.$$

Let $b_n = 20^n/n!$. The Ratio Test gives

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{20^{n+1}}{(n+1)!} \frac{n!}{20^n} = \frac{20}{n+1}, \\ \lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} &= 0 < 1. \end{aligned}$$

Thus the absolute-value series converges, so the original series converges absolutely.

Therefore, The series converges absolutely.

Question 2. Find the first three nonzero terms of the Maclaurin series for the following function and the values of x for which the series converges absolutely.

$$f(x) = (1 - 3x + 2x^2)e^x.$$

Solution

The Maclaurin formula is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Put $P(x) = 1 - 3x + 2x^2$, so $f(x) = P(x)e^x$. The product rule gives

$$\begin{aligned} f'(x) &= (P(x) + P'(x))e^x, \\ f''(x) &= (P(x) + 2P'(x) + P''(x))e^x. \end{aligned}$$

Since $P(0) = 1$, $P'(0) = -3$, and $P''(0) = 4$,

$$\begin{aligned} f(0) &= 1, \\ f'(0) &= 1 - 3 = -2, \\ f''(0) &= 1 + 2(-3) + 4 = -1. \end{aligned}$$

Substituting the coefficients, we obtain

$$1 - 2x + \frac{-1}{2!}x^2 + \cdots = \boxed{1 - 2x - \frac{1}{2}x^2 + \cdots}.$$

These are the first three nonzero terms.

Since the exponential series converges absolutely for every real x , multiplying it by a polynomial preserves absolute convergence for every $x \in \mathbb{R}$.

Therefore, the series converges absolutely for $x \in \mathbb{R}$.