

Variant A — Question 1

Determine whether

$$\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(2n)}{n^2 + 4}$$

converges absolutely, converges conditionally, or diverges. Here $\tan^{-1}(2n)$ means $\arctan(2n)$, the inverse tangent function.

Step 1: Form the absolute-value series. For every $n \geq 1$, $2n > 0$, so $\arctan(2n) > 0$. Also $|(-1)^n| = 1$. Thus

$$\left| (-1)^n \frac{\arctan(2n)}{n^2 + 4} \right| = \frac{\arctan(2n)}{n^2 + 4}.$$

We therefore investigate $\sum_{n=1}^{\infty} \frac{\arctan(2n)}{n^2 + 4}$.

Step 2: Find a convergent upper bound. For every positive real number t , $0 < \arctan(t) < \pi/2$. Since $n^2 + 4 \geq n^2$, we obtain

$$0 \leq \frac{\arctan(2n)}{n^2 + 4} \leq \frac{\pi}{2(n^2 + 4)} \leq \frac{\pi}{2n^2}.$$

The comparison series

$$\sum_{n=1}^{\infty} \frac{\pi}{2n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges because it is a constant multiple of a p -series with $p = 2 > 1$.

Step 3: Apply the comparison test and classify. The terms of the absolute-value series are nonnegative and bounded above by the terms of a convergent series. Therefore, the direct comparison test gives

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\arctan(2n)}{n^2 + 4} \right| < \infty.$$

The given series converges absolutely.

Absolute convergence also guarantees convergence of the original series. An alternating-series test is unnecessary here because absolute convergence already settles the classification.

Variant A — Question 2

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}}.$$

Step 1: Identify the center and apply the ratio test. The series is centered at $x = -1$. Let $a_n = (x+1)^n/\sqrt{n}$. For $x \neq -1$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x+1)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(x+1)^n} \right| = |x+1| \sqrt{\frac{n}{n+1}}.$$

Since $n/(n+1) \rightarrow 1$,

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x+1|.$$

At $x = -1$, every term is zero, so the series converges there as well.

Step 2: Solve the convergence inequality. The ratio test guarantees absolute convergence when

$$|x+1| < 1 \iff -1 < x+1 < 1 \iff -2 < x < 0.$$

It gives divergence when $|x+1| > 1$. The radius is therefore $\boxed{R=1}$. At the endpoints $x = -2$ and $x = 0$, the ratio-test limit is 1, so each endpoint must be checked separately.

Step 3: Check the left endpoint $x = -2$. Substitution gives $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$. Set $b_n = 1/\sqrt{n}$.

Then $b_n > 0$, and

$$n+1 > n \implies \sqrt{n+1} > \sqrt{n} \implies b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n.$$

Also $\lim_{n \rightarrow \infty} b_n = 0$. Hence the alternating-series test proves convergence. However, $\sum 1/\sqrt{n} = \sum 1/n^{1/2}$ diverges because $p = 1/2 \leq 1$. Thus convergence at $x = -2$ is conditional, and this endpoint is included.

Step 4: Check the right endpoint $x = 0$. The series becomes $\sum_{n=1}^{\infty} 1/\sqrt{n}$, the divergent p -series just considered. This endpoint is excluded.

$$\boxed{R=1, \quad \text{interval of convergence } [-2, 0).}$$

Variant B — Question 1

Determine whether

$$\sum_{n=1}^{\infty} (-1)^n \frac{(n+2)^n}{(4n)^n}$$

converges absolutely, converges conditionally, or diverges.

Step 1: Form the absolute-value series. Since $(n+2)^n$ and $(4n)^n$ are positive for $n \geq 1$,

$$\left| (-1)^n \frac{(n+2)^n}{(4n)^n} \right| = \frac{(n+2)^n}{(4n)^n} = \left(\frac{n+2}{4n} \right)^n.$$

Thus we investigate $\sum_{n=1}^{\infty} \left(\frac{n+2}{4n} \right)^n$.

Step 2: Apply the root test. The n th powers make the root test convenient. If $c_n = ((n+2)/(4n))^n$, then

$$\sqrt[n]{c_n} = \frac{n+2}{4n} = \frac{1}{4} + \frac{1}{2n}.$$

Taking the limit gives

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{c_n} = \frac{1}{4} + 0 = \frac{1}{4}.$$

Step 3: Interpret the result. The root test states that a series converges absolutely if the limit of the n th root of the absolute value of its n th term is less than 1. Here $L = 1/4 < 1$. Therefore,

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{(n+2)^n}{(4n)^n} \right|$$

converges, and

The given series converges absolutely.

The alternating sign does not change this conclusion: the positive absolute-value series already converges.

Variant B — Question 2

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(3x+1)^n}{n}.$$

Step 1: Identify the center and apply the ratio test. Write $3x+1 = 3(x+1/3)$; the center is $x = -1/3$. Let $a_n = (3x+1)^n/n$. For $x \neq -1/3$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(3x+1)^{n+1}}{n+1} \frac{n}{(3x+1)^n} \right| = |3x+1| \frac{n}{n+1}.$$

Since $n/(n+1) \rightarrow 1$, the ratio-test limit is $L = |3x+1|$. At $x = -1/3$, every term is zero, so the series converges there.

Step 2: Solve the convergence inequality and find the radius. Absolute convergence holds when

$$|3x+1| < 1 \iff -1 < 3x+1 < 1 \iff -2 < 3x < 0 \iff -\frac{2}{3} < x < 0.$$

Equivalently, $|x+1/3| < 1/3$, so $\boxed{R = 1/3}$. The series diverges when $|3x+1| > 1$. Check the two endpoints separately because the ratio-test limit is 1 there.

Step 3: Check the left endpoint $x = -2/3$. Here $3x+1 = -1$, so the series becomes $\sum_{n=1}^{\infty} (-1)^n/n$.

Let $b_n = 1/n$. Then $b_n > 0$, and

$$n+1 > n \implies b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n, \quad \lim_{n \rightarrow \infty} b_n = 0.$$

The alternating-series test therefore proves convergence. The absolute-value series $\sum 1/n$ is the divergent harmonic series. Hence the convergence is conditional and $x = -2/3$ is included.

Step 4: Check the right endpoint $x = 0$. Here $3x+1 = 1$, so the series becomes the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges. Thus $x = 0$ is excluded.

$$\boxed{R = \frac{1}{3}, \quad \text{interval of convergence } \left[-\frac{2}{3}, 0\right)}.$$