

Variant A – Question 1

Determine whether

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

converges absolutely, converges conditionally, or diverges.

Step 1: Identify the alternating form. Write the series as $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$, where $b_n = 1/\sqrt{n} > 0$. The factor $(-1)^{n+1}$ alternates between $+1$ and -1 .

Step 2: Show that the positive terms decrease. For every $n \geq 1$,

$$n+1 > n \implies \sqrt{n+1} > \sqrt{n} \implies \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}.$$

Taking reciprocals reverses the inequality because both quantities are positive. Thus $b_{n+1} < b_n$.

Step 3: Check that the positive terms tend to zero. Since $\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

The positive terms decrease and tend to zero, so the alternating-series test proves that the original series converges.

Step 4: Check absolute convergence. Since $|(-1)^{n+1}| = 1$, the absolute-value series is

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}.$$

A p -series $\sum 1/n^p$ converges only when $p > 1$. Here $p = 1/2 \leq 1$, so the absolute-value series diverges.

Conclusion. The original series converges, but its absolute-value series diverges. Therefore,

The series converges conditionally.

Variant A – Question 2

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}.$$

Step 1: Identify the center and apply the ratio test. The center is $x = -1$. Let $a_n = n(x+1)^n/4^n$. For $x \neq -1$,

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)(x+1)^{n+1}}{4^{n+1}} \frac{4^n}{n(x+1)^n} \right| \\ &= \frac{n+1}{n} \frac{|x+1|}{4}. \end{aligned}$$

Because $(n+1)/n = 1 + 1/n \rightarrow 1$,

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x+1|}{4}.$$

At $x = -1$, every term is zero, so the series converges there too.

Step 2: Solve the convergence inequality. Absolute convergence holds when

$$\frac{|x+1|}{4} < 1 \iff |x+1| < 4 \iff -4 < x+1 < 4 \iff -5 < x < 3.$$

Hence $\boxed{R = 4}$. The ratio test gives divergence outside this interval. At $x = -5$ and $x = 3$, its limit is 1, so check both endpoints directly.

Step 3: Check the left endpoint $x = -5$. Here $x+1 = -4$, and the series becomes

$$\sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} n(-1)^n.$$

The terms do not tend to zero: their magnitudes are $n \rightarrow \infty$ (even-indexed terms are positive and odd-indexed terms are negative). The necessary condition for convergence fails, so this endpoint is excluded.

Step 4: Check the right endpoint $x = 3$. Here $x+1 = 4$, so the series becomes $\sum_{n=1}^{\infty} n$. Its terms tend to infinity, rather than zero. It therefore diverges, and this endpoint is excluded.

$$\boxed{R = 4, \quad \text{interval of convergence } (-5, 3).}$$

Variant B – Question 1

Determine whether

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$$

converges absolutely, converges conditionally, or diverges.

Step 1: Identify the positive terms. Let $b_n = n/(n^2 + 1) > 0$ for $n \geq 1$. The original series is $\sum (-1)^{n+1} b_n$.

Step 2: Prove that $b_{n+1} < b_n$. Subtract the two consecutive terms and use a common denominator:

$$\begin{aligned} b_n - b_{n+1} &= \frac{n}{n^2 + 1} - \frac{n+1}{(n+1)^2 + 1} \\ &= \frac{n(n^2 + 2n + 2) - (n+1)(n^2 + 1)}{(n^2 + 1)(n^2 + 2n + 2)} \\ &= \frac{(n^3 + 2n^2 + 2n) - (n^3 + n^2 + n + 1)}{(n^2 + 1)(n^2 + 2n + 2)} \\ &= \frac{n^2 + n - 1}{(n^2 + 1)(n^2 + 2n + 2)} > 0. \end{aligned}$$

Indeed, the denominator is positive and, for $n \geq 1$, $n^2 + n - 1 \geq 1$. Therefore $b_n > b_{n+1}$.

Step 3: Check the zero limit and apply the alternating-series test. Dividing the numerator and denominator by n^2 gives

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 + 1/n^2} = 0.$$

Since b_n decreases to zero, the alternating-series test proves convergence of the original series.

Step 4: Show that the absolute-value series diverges. The absolute-value series is $\sum n/(n^2 + 1)$. Compare it with the harmonic series $\sum 1/n$. Since $n^2 + 1 \leq 2n^2$ for $n \geq 1$,

$$\frac{n}{n^2 + 1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The series $\sum 1/(2n)$ diverges, so the direct comparison test proves divergence of $\sum n/(n^2 + 1)$.

The original series converges conditionally.

Variant B – Question 2

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$

Step 1: Identify the center and apply the ratio test. The series is centered at $x = 2$. Let $a_n = (x-2)^n/(n3^n)$. For $x \neq 2$,

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \frac{n3^n}{(x-2)^n} \right| \\ &= \frac{|x-2|}{3} \frac{n}{n+1}. \end{aligned}$$

Since $n/(n+1) \rightarrow 1$, the limit is

$$L = \frac{|x-2|}{3}.$$

At $x = 2$, all terms are zero, so convergence also holds at the center.

Step 2: Solve the convergence inequality. The ratio test guarantees absolute convergence when

$$\frac{|x-2|}{3} < 1 \iff |x-2| < 3 \iff -3 < x-2 < 3 \iff -1 < x < 5.$$

Thus $\boxed{R=3}$. The series diverges outside this interval. At the endpoints the ratio-test limit equals 1, so additional tests are required.

Step 3: Check the left endpoint $x = -1$. Here $x-2 = -3$, so the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

For $b_n = 1/n$, we have $b_n > 0$, $\lim_{n \rightarrow \infty} b_n = 0$, and

$$n+1 > n \implies \frac{1}{n+1} < \frac{1}{n} \implies b_{n+1} < b_n.$$

The alternating-series test proves convergence. Its absolute-value series is the divergent harmonic series $\sum 1/n$, so convergence is conditional. Include $x = -1$.

Step 4: Check the right endpoint $x = 5$. Here $x-2 = 3$, and the series becomes $\sum 3^n/(n3^n) = \sum 1/n$. This harmonic series diverges, so exclude $x = 5$.

$$\boxed{R=3, \quad \text{interval of convergence } [-1, 5).}$$