

Calculus II

Quiz 1

Variant A

Name _____

CRN _____

1. Determine whether the following sequence $\{a_n\}$ converges or diverges. If it converges, find its limit.

$$a_n = \sqrt{n^2 + 4n} - n$$

(1.5 point)

2. Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

(1.5 point)

Calculus II

Quiz 1

Variant B

Name _____

CRN _____

1. Determine whether the following sequence $\{a_n\}$ converges or diverges. If it converges, find its limit.

$$a_n = \ln(3n + 5) - \ln(n + 2)$$

(1.5 point)

2. Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

(1.5 point)

Calculus II — Quiz 1 Solutions

Comprehensive Solution Sheet

Variant B Solutions

Question 1 (1.5 points)

Problem: Determine whether the sequence $\{a_n\}$ converges or diverges. If it converges, find its limit.

$$a_n = \ln(3n + 5) - \ln(n + 2)$$

Detailed Solution:

- **Logarithm Properties:** Using $\ln(A) - \ln(B) = \ln(\frac{A}{B})$, rewrite $a_n = \ln(\frac{3n+5}{n+2})$.
- **Evaluate Limit:** $\lim_{n \rightarrow \infty} a_n = \ln(\lim_{n \rightarrow \infty} \frac{3n+5}{n+2})$ due to the continuity of $\ln(x)$.
- **Inner Limit:** Dividing numerator and denominator by n yields $\lim_{n \rightarrow \infty} \frac{3 + 5/n}{1 + 2/n} = \frac{3+0}{1+0} = 3$.
- **Conclusion:** $\lim_{n \rightarrow \infty} a_n = \ln(3)$. The sequence **converges** to $\ln(3)$.

Point Breakdown (1.5 points total):

- **0.5 pts:** Combining/simplifying terms into a single rational fraction inside the logarithm using log laws.
- **0.5 pts:** Setting up the limit and evaluating the internal rational expression to 3.
- **0.5 pts:** Stating the final limit value $\ln(3)$ and concluding convergence.

Question 2 (1.5 points)

Problem: Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

Detailed Solution:

- **Integral Test:** The function $f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing for $x \geq 2$.
- **Improper Integral Setup:** $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx$.
- **u -Substitution:** Let $u = \ln x$, $du = \frac{1}{x} dx$. Limits change from $\ln 2$ to $\ln t$. Integral becomes $\lim_{t \rightarrow \infty} [-\frac{1}{u}]_{\ln 2}^{\ln t}$.
- **Evaluation & Conclusion:** Evaluating yields $0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$. The integral converges, so the series **converges**.

Point Breakdown (1.5 points total):

- **0.5 pts:** Selecting the Integral Test and defining the corresponding improper integral correctly.
- **0.5 pts:** Executing u -substitution and finding the correct antiderivative $-\frac{1}{\ln x}$.
- **0.5 pts:** Evaluating limits to a finite value ($\frac{1}{\ln 2}$) and stating convergence.

Variant A Solutions

Question 1 (1.5 points)

Problem: Determine whether the sequence $\{a_n\}$ converges or diverges. If it converges, find its limit.

$$a_n = \sqrt[n^2 + 4n]{n^2 + 4n} - n$$

Detailed Solution:

- **Rationalize:** Multiply and divide by the conjugate $(\sqrt[n^2 + 4n]{n^2 + 4n} + n)$ to resolve the $\infty - \infty$ form.
- **Algebraic Simplification:** $a_n = \frac{(n^2 + 4n) - n^2}{\sqrt[n^2 + 4n]{n^2 + 4n} + n} = \frac{4n}{\sqrt[n^2 + 4n]{n^2 + 4n} + n}$.
- **Evaluate Limit:** Divide numerator and denominator by n (noting $\sqrt[n^2]{n^2} = n$ in the denominator): $\lim_{n \rightarrow \infty} \frac{4}{\sqrt[1 + 4/n]{1 + 1}}$.
- **Conclusion:** $\frac{4}{\sqrt{1 + 1}} = \frac{4}{2} = 2$. The sequence **converges** to 2.

Point Breakdown (1.5 points total):

- **0.5 pts:** Multiplying and dividing by the proper conjugate expression.
- **0.5 pts:** Simplifying algebraically to $\frac{4n}{\sqrt[n^2 + 4n]{n^2 + 4n} + n}$ and dividing terms by n .
- **0.5 pts:** Computing the final limit value as 2 and concluding convergence.

Question 2 (1.5 points)

Problem: Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

Detailed Solution:

- **Integral Test:** The function $f(x) = \frac{1}{x \ln x}$ is continuous, positive, and decreasing for $x \geq 2$.
- **Improper Integral Setup:** $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx$.
- **u-Substitution:** Let $u = \ln x$, $du = \frac{1}{x} dx$. Limits change from $\ln 2$ to $\ln t$. Integral becomes $\lim_{t \rightarrow \infty} [\ln|u|]_{\ln 2}^{\ln t}$.
- **Evaluation & Conclusion:** Yields $\lim_{t \rightarrow \infty} (\ln(\ln t) - \ln(\ln 2)) = \infty$. Since the integral diverges, the series **diverges**.

Point Breakdown (1.5 points total):

- **0.5 pts:** Correctly setting up the Integral Test and corresponding improper integral.
- **0.5 pts:** Evaluating the antiderivative properly using substitution $(\ln|\ln x|)$.
- **0.5 pts:** Determining that the limit goes to infinity and stating that the series diverges.