

Calculus II

Quiz 1

Variant A

Name _____

CRN _____

1. Determine whether each of the following sequences $\{a_n\}$ converges or diverges. If it converges, find its limit.

a) $a_n = \frac{3^n}{n!}$

b) $a_n = \frac{\sin(4n)}{n}$

(1.5 point)

2. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2+2n}{\sqrt{n^7+n^3}}$$

(1.5 point)

Calculus II

Quiz 1

Variant B

Name _____

CRN _____

1. Determine whether each of the following sequences $\{a_n\}$ converges or diverges. If it converges, find its limit.
 - a) $a_n = e^{-n} \cos(n)$
 - b) $a_n = \frac{2^n}{n^2}$

(1.5 point)

2. Determine whether the following series converges or diverges.
 $\sum_{n=1}^{\infty} \frac{1}{5^{n-n}}$

(1.5 point)

Calculus II — Quiz 1 Solutions

Comprehensive Solution Sheet (Set 2)

Variant A Solutions

Question 1 (1.5 points total — 0.75 pts each part)

Problem: Determine whether each sequence $\{a_n\}$ converges or diverges. Find its limit if it converges.

Part (a): $a_n = \frac{3^n}{n!}$

- **Analysis:** For large n , the factorial growth in the denominator ($n!$) vastly outperforms the exponential growth in the numerator (3^n).
- **Limit Evaluation:** $\lim_{n \rightarrow \infty} \frac{3^n}{n!} = 0$ (by growth rates or by bounding via $0 < \frac{3^n}{n!} \leq \frac{9}{n}$ for $n \geq 3$ and applying the Squeeze Theorem).
- **Conclusion:** The sequence **converges** to 0. (0.75 pts)

Part (b): $a_n = \frac{\sin(4n)}{n}$

- **Analysis:** Since $-1 \leq \sin(4n) \leq 1$ for all n , we can bound the sequence: $-\frac{1}{n} \leq \frac{\sin(4n)}{n} \leq \frac{1}{n}$.
- **Squeeze Theorem:** Taking limits as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} (-\frac{1}{n}) = 0$ and $\lim_{n \rightarrow \infty} (\frac{1}{n}) = 0$. Thus, by the Squeeze Theorem, $\lim_{n \rightarrow \infty} \frac{\sin(4n)}{n} = 0$.
- **Conclusion:** The sequence **converges** to 0. (0.75 pts)

Question 2 (1.5 points)

Problem: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2 + 2n}{\sqrt{n^7 + n^3}}$ converges or diverges.

- **Behavior Analysis (Limit Comparison Test):** For large n , the dominant term in the numerator is n^2 , and in the denominator is $\sqrt{n^7} = n^{7/2}$.
- **Estimate Order:** $\frac{n^2}{n^{7/2}} = \frac{1}{n^{7/2-2}} = \frac{1}{n^{3/2}}$.
- **Comparison Series:** Compare with the p-series $\sum \frac{1}{n^{3/2}}$, which **converges** because $p = \frac{3}{2} > 1$.
- **Limit Comparison:** $\lim_{n \rightarrow \infty} \left[\frac{n^2 + 2n}{\sqrt{n^7 + n^3}} / \frac{1}{n^{3/2}} \right] = \lim_{n \rightarrow \infty} \frac{n^{7/2} + 2n^{5/2}}{\sqrt{n^7 + n^3}} = 1$ (finite and positive).
- **Conclusion:** By the Limit Comparison Test, since the comparison p-series converges, the given series **converges**.

Point Breakdown (Question 2: 1.5 points total):

- **0.5 pts:** Determining the dominant behavior/asymptotic order ($\frac{1}{n^{3/2}}$).
- **0.5 pts:** Selecting an appropriate test (Limit Comparison Test) with a known convergent p-series ($p = 3/2$).
- **0.5 pts:** Computing the limit comparison ratio correctly and stating the final conclusion of convergence.

Variant B Solutions

Question 1 (1.5 points total — 0.75 pts each part)

Problem: Determine whether each sequence $\{a_n\}$ converges or diverges. Find its limit if it converges.

Part (a): $a_n = e^{-n} \cos(n)$

- **Analysis:** We know that $-1 \leq \cos(n) \leq 1$ for all integers n . Multiplying by e^{-n} (which is > 0): $-e^{-n} \leq e^{-n} \cos(n) \leq e^{-n}$.
- **Squeeze Theorem:** As $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} (-e^{-n}) = 0$ and $\lim_{n \rightarrow \infty} (e^{-n}) = 0$. By the Squeeze Theorem, the limit of the middle expression is also 0.
- **Conclusion:** The sequence **converges** to 0. (0.75 pts)

Part (b): $a_n = \frac{2^n}{n^2}$

- **Analysis:** Exponential growth in the numerator (2^n) vastly outpaces polynomial growth in the denominator (n^2) as $n \rightarrow \infty$.
- **Limit Evaluation:** $\lim_{n \rightarrow \infty} \frac{2^n}{n^2} = \infty$ (can be verified by applying L'Hôpital's Rule twice on the continuous extension).
- **Conclusion:** The sequence **diverges** (tends to infinity). (0.75 pts)

Question 2 (1.5 points)

Problem: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{5^{n-1}}$ converges or diverges.

- **Series Identification:** Rewrite the series in standard geometric form: $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1}$.
- **Common Ratio:** This is a geometric series with common ratio $|r| = \frac{1}{5} < 1$.
- **Convergence Test:** By the Geometric Series Test, any geometric series with $|r| < 1$ converges.
- **Conclusion:** The series **converges**.

Point Breakdown (Question 2: 1.5 points total):

- **0.5 pts:** Recognizing the series type as a geometric series and identifying the common ratio r .
- **0.5 pts:** Stating the condition for geometric series convergence ($|r| < 1$).
- **0.5 pts:** Formulating the correct final conclusion that the series converges.